

# Noncommutative Superspaces Covariant Under $OSp_q(1|2)$ Algebra

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## Abstract

Using the corepresentation of the quantum group  $SL_q(2)$  a general method for constructing noncommutative spaces covariant under its coaction is developed. The method allows us to treat the quantum plane and Podleś' quantum spheres in a unified way and to construct higher dimensional noncommutative spaces systematically. Furthermore, we extend the method to the quantum supergroup  $OSp_q(1|2)$ . In particular, a one-parameter family of covariant algebras, which may be interpreted as noncommutative superspheres, is constructed.

## 1 Introduction

Quantum groups provide a very powerful tool for investigations of noncommutative geometry, since they may be regarded as a noncommutative extension of linear Lie groups. Pioneering works by Manin [1], Woronowicz [2], Wess and Zumino [3] are followed by hundreds of publications (see for example [4, 5] and references therein). One way of using quantum groups for noncommutative geometry is, by regarding them as an example of noncommutative manifolds, to develop a harmonic analysis on quantum groups. Another way, which may be more familiar to large class of physicists, is to regard a quantum group as a transformation matrix of vectors. Because of the noncommutative nature of quantum groups, the vectors transformed by a quantum group are *a priori* noncommutative. Namely, the components of the vectors have nontrivial commutation relations. In order to fit such vectors in theories of physics, we require covariance, that is, the commutation relations are preserved by quantum group transformations. Noncommutative vectors obeying a covariant algebra may be given geometrical interpretation.

Let us consider  $SU_q(2)$  as an example. If we consider a covariant algebra transformed by the fundamental representation, it may be interpreted as a noncommutative analogue of two dimensional flat space [1]. While if we take a

covariant algebra for adjoint representation, it may be regarded as a noncommutative extension of 3-sphere [6]. Higher dimensional representations will give higher dimensional noncommutative spaces. However, no such work has been done because mainly of computational difficulty. Furthermore, there are only a few works on noncommutative analogues of superspaces in the context of quantum groups despite the fact that supersymmetry is one of the most important notions of theoretical physics.

In the present work, using the corepresentations of quantum groups, we develop a general method for constructing noncommutative spaces for the simplest and the most important quantum (super) groups  $SL_q(2)$  and  $OSp_q(1|2)$ . In the first part of this paper (§2 and §3), the case  $SL_q(2)$  is considered. It will be seen that, by our method, the quantum plane and the quantum spheres are treated on the same footing and that the higher dimensional noncommutative spaces may be constructed systematically. In the second part (§4), we extend the method to  $OSp_q(1|2)$ . As an application of our method, noncommutative superspace and a one-parameter family of noncommutative superspheres are explicitly constructed. Finally §5 is devoted to concluding remarks.

## 2 $SL_q(2)$ and its corepresentations

This section is a brief review of the definitions and representations for the quantum group  $SL_q(2)$  and the quantum algebra  $U_q[sl(2)]$  that is dual to  $SL_q(2)$ . There are several good textbooks on this topics. Readers may refer, for example, to [4, 5] and references therein.

The quantum group  $SL_q(2)$  is generated by four elements  $a, b, c$  and  $d$  subject to the relations

$$\begin{aligned} ab &= qba, & ac &= qca, & bd &= qdb, \\ cd &= qdc, & bc &= cb, & ad - da &= (q - q^{-1})bc, \\ ad - qbc &= da - q^{-1}bc = 1. \end{aligned} \quad (1)$$

As is well-known, the coproduct ( $\Delta$ ), the counit ( $\epsilon$ ) and the antipode ( $S$ ) defined as follows make  $SL_q(2)$  a Hopf algebra:

$$\begin{aligned} \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, & \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ S \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \end{aligned} \quad (2)$$

With the Hopf algebra mappings, we define a corepresentation of a quantum group. A vector space  $V$  is called a right  $SL_q(2)$ -comodule if there exists a linear mapping  $\varphi_R : V \rightarrow V \otimes SL_q(2)$  satisfying

$$(\varphi_R \otimes \text{id}) \circ \varphi_R = (\text{id} \otimes \Delta) \circ \varphi_R, \quad (\text{id} \otimes \epsilon) \circ \varphi_R = \text{id}. \quad (3)$$

Similarly, the left  $SL_q(2)$ -comodule is defined as a vector space  $V$  equipped with a linear mapping  $\varphi_L : V \rightarrow SL_q(2) \otimes V$  such that

$$(\text{id} \otimes \varphi_L) \circ \varphi_L = (\Delta \otimes \text{id}) \circ \varphi_L, \quad (\epsilon \otimes \text{id}) \circ \varphi_L = \text{id}. \quad (4)$$

The mapping  $\varphi_R$  ( $\varphi_L$ ) is called a corepresentation, or, equivalently, a right (left) coaction of  $SL_q(2)$  on  $V$ . It is known that each irreducible corepresentation of  $SL_q(2)$  is, as classical  $SL(2)$ , specified by the highest weight  $j$  which takes any nonnegative integral or half-integral values. Let  $V^{(j)}$  be a right  $SL_q(2)$ -comodule with the highest weight  $j$  and  $\{e_m^j, m = j, j-1, \dots, -j\}$  be its basis:

$$\varphi_R(e_m^j) = \sum_{m'} e_{m'}^j \otimes T_{m'm}^{(j)}, \quad T_{m'm}^{(j)} \in SL_q(2) \quad (5)$$

The corepresentations of  $SL_q(2)$  have been obtained explicitly [7–9]. We here give  $j = 1/2$  and  $j = 1$  corepresentation matrices as an example:

$$T^{(1/2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (6)$$

$$T^{(1)} = \begin{pmatrix} a^2 & (1+q^{-2})^{1/2}ab & b^2 \\ (1+q^{-2})^{1/2}ac & 1+(q+q^{-1})bc & (1+q^{-2})^{1/2}bd \\ c^2 & (1+q^{-2})^{1/2}cd & d^2 \end{pmatrix}. \quad (7)$$

A comodule of a quantum group is, in general, a module *i.e.* a representation space of the dual quantum algebra. We define the action of  $U_q[sl(2)]$  on  $V^{(j)}$  by

$$Xe_m^j = ((\text{id} \otimes X) \circ \varphi_R)(e_m^j) = \sum_{m'} e_{m'}^j \langle X, T_{m'm}^{(j)} \rangle, \quad X \in U_q[sl(2)] \quad (8)$$

where  $\langle \cdot, \cdot \rangle : U_q[sl(2)] \otimes SL_q(2) \rightarrow \mathbb{C}$  is the duality pairing of two Hopf algebras. Then it may be verified that the matrix  $\langle X, T_{m'm}^{(j)} \rangle$  gives an irreducible representation of  $U_q[sl(2)]$  with the highest weight  $j$ . The product space  $V^{(j_1)} \otimes V^{(j_2)}$  is, in general, reducible and is decomposed into irreducible spaces as

$$j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \dots \oplus |j_1 - j_2|. \quad (9)$$

The decomposition is carried out by the Clebsch-Gordan coefficients (CGC)

$$e_M^J(j_1, j_2) = \sum_{m_1, m_2} C_{m_1 m_2 M}^{j_1 j_2 J} e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}. \quad (10)$$

The CGC satisfy the following orthogonality relations

$$\sum_{j, m} C_{m_1 m_2 m}^{j_1 j_2 j} C_{m'_1 m'_2 m'}^{j_1 j_2 j} = \delta_{m_1 m'_1} \delta_{m_2 m'_2}, \quad (11)$$

$$\sum_{m_1, m_2} C_{m_1 m_2 m}^{j_1 j_2 j} C_{m_1 m_2 m'}^{j_1 j_2 j'} = \delta_{jj'} \delta_{mm'}. \quad (12)$$

Two corepresentations are also coupled by CGC. The coupling is given by the formula that is called the Wigner's product law [10]:

$$\delta_{jj'} T_{mm'}^{(j)} = \sum_{\substack{m_1, m_2 \\ m'_1, m'_2}} C_{m_1 m_2 m}^{j_1 j_2 j} C_{m'_1 m'_2 m'}^{j_1 j_2 j'} T_{m_1 m'_1}^{(j_1)} T_{m_2 m'_2}^{(j_2)}. \quad (13)$$

### 3 Covariant algebras of $SL_q(2)$

#### 3.1 General prescription

In this section, we give a general prescription to construct  $SL_q(2)$ -covariant algebras. By covariant algebras, we mean algebras whose defining relations are preserved under the right coaction of  $SL_q(2)$  defined by (5). Probably, the simplest way to find such an algebra is to introduce an algebraic structure on the comodule  $V^{(j)}$ . Let  $\mu$  be a product in  $V^{(j)}$ , i.e.,  $\mu(f \otimes g) = fg$ ,  $f, g \in V^{(j)}$ . We specifically consider the following composite object

$$\mu(e_M^J(j, j)) = \sum_{m_1, m_2} C_{m_1 m_2 M}^{jjJ} e_{m_1}^j e_{m_2}^j. \quad (14)$$

The right coaction on (14) is shown to be

$$\varphi_R \circ \mu(e_M^J(j, j)) = \sum_{M'} \mu(e_{M'}^J(j, j)) \otimes T_{M'M}^{(J)}. \quad (15)$$

The proof may be done in a straightforward way by inverting the relation (14)

$$e_{m_1}^j e_{m_2}^j = \sum_{JM} C_{m_1 m_2 M}^{jjJ} \mu(e_M^J(j, j)), \quad (16)$$

and subsequently using the product law (13)

$$\begin{aligned} \varphi_R \circ \mu(e_M^J(j, j)) &= \sum_{m_1 m_2} C_{m_1 m_2 M}^{jjJ} \varphi_R(e_{m_1}^j) \varphi_R(e_{m_2}^j) \\ &= \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} C_{m_1 m_2 M}^{jjJ} e_{m'_1}^j e_{m'_2}^j \otimes T_{m'_1 m_1}^{(j)} T_{m'_2 m_2}^{(j)} \\ &\stackrel{(16)}{=} \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} \sum_{J' M'} C_{m_1 m_2 M}^{jjJ} C_{m'_1 m'_2 M'}^{jjJ'} \mu(e_{M'}^{J'}(j, j)) \otimes T_{m'_1 m_1}^{(j)} T_{m'_2 m_2}^{(j)} \\ &\stackrel{(13)}{=} \sum_{M'} \mu(e_{M'}^J(j, j)) \otimes T_{M'M}^{(J)}. \end{aligned}$$

Employing (15) we now extract a set of covariant relations under  $\varphi_R$ . The  $J = 0$  relation  $\varphi_R \circ \mu(e_0^0(j, j)) = \mu(e_0^0(j, j))$  signifies that  $\mu(e_0^0(j, j))$  is a scalar under the right coaction. It may be equated to a constant parameter  $r$

$$\mu(e_0^0(j, j)) = \sum_{m_1, m_2} C_{m_1 m_2 0}^{jj0} e_{m_1}^j e_{m_2}^j = r. \quad (17)$$

If  $J = j$ , then  $\mu(e_m^j(j, j))$  and  $e_m^j$  transform identically under  $\varphi_R$ . Therefore  $\mu(e_m^j(j, j))$  is, in general, proportional to  $e_m^j$ . It may be noted that the following relations are covariant

$$\mu(e_m^j(j, j)) = \sum_{m_1, m_2} C_{m_1 m_2 m}^{jjj} e_{m_1}^j e_{m_2}^j = \xi e_m^j, \quad (18)$$

where the proportionality constant  $\xi \rightarrow 0$  as  $q \rightarrow 1$ . For  $J \neq 0, j$ , the element  $\mu(e_M^J(j, j))$  can not be a scalar, nor proportional to  $e_M^J$  as they transform differently. The relevant covariant relations are, therefore, of the form

$$\mu(e_M^J(j, j)) = \sum_{m_1, m_2} C_{m_1 m_2 M}^{jjJ} e_{m_1}^j e_{m_2}^j = 0. \quad (19)$$

As will be seen from the examples given in the next subsection, the simultaneous use of all relations from (17) to (19) gives an inconsistent result, since some of them do not have correct classical limits. In order to obtain a consistent covariant algebra, we have to make a choice regarding the relations to be used for defining the algebra. Then the consistency has to be verified. As it is clear from the above discussion, the covariant algebras can have at most two more parameters ( $r, \xi$ ) in addition to the deformation parameter  $q$ . It is emphasised that the origin of the parameters is clearly explained in the framework of the representation theory. We have formulated a method to construct  $SL_q(2)$ -covariant algebras with respect to the right coaction. It is possible to repeat the same discussion for the left coaction.

### 3.2 Quantum plane and quantum spheres

We apply the general prescription in the previous subsection to  $j = 1/2$  and  $j = 1$  corepresentations. As will be seen, the obtained covariant algebras correspond to the quantum plane of Manin for  $j = 1/2$  and the quantum spheres of Podleś for  $j = 1$ .

Let us start with  $j = 1/2$  case where the relevant tensor product decomposition is given by  $1/2 \otimes 1/2 = 1 \oplus 0$ . We denote the basis of  $V^{(1/2)}$  by  $(x, y) = (e_{1/2}^{1/2}, e_{-1/2}^{1/2})$ . The quantum matrix which coacts on this basis is given by (6). Using explicit formula of CGC given in [4, 5], we obtain from (17) for  $J = 0$

$$xy - qyx = r. \quad (20)$$

If we set  $r = 0$ , then (20) is reduced to the quantum plane relation. For  $J = 1$ , we obtain, from (19), unacceptable relations such as  $x^2 = y^2 = 0$ . Thus we take only (20) as defining relations of our covariant algebra.

We next investigate  $j = 1$  case, namely, the adjoint corepresentation of  $SL_q(2)$ . Since the adjoint corepresentation of the classical  $SL(2)$  corresponds to the fundamental corepresentation of  $SO(3)$ , the covariant algebra may be interpreted as a sphere. The relevant tensor product decomposition is  $1 \otimes 1 = 2 \oplus 1 \oplus 0$ . The basis of  $V^{(1)}$ , on which the quantum matrix (7) coacts, is denoted by  $x_m = e_m^1$ . The covariant relation for  $J = 0$  is obtained via (17)

$$x_0^2 - qx_1x_{-1} - q^{-1}x_{-1}x_1 = r. \quad (21)$$

Explicit constructions for the  $J = 1$  case are obtained via (18)

$$\begin{aligned} (1 - q^2)x_0^2 + qx_{-1}x_1 - qx_1x_{-1} &= \xi x_0, \\ x_{-1}x_0 - q^2x_0x_{-1} &= \xi x_{-1}, \quad x_0x_1 - q^2x_1x_0 = \xi x_1. \end{aligned} \quad (22)$$

For  $J = 2$ , we obtain, from (19), unacceptable relations such as  $x_{\pm 1}^2 = 0$ . Thus we take (21) and (22) as defining relations of our covariant algebra. We need to check the following conditions in order to verify whether or not the algebra is well-defined:

- (a) The constant  $r$  commutes with all generators
- (b) Product of three generators, say  $x_1 x_0 x_{-1}$ , has two ways of reversing its ordering:

$$\begin{array}{ccccc}
 & & x_1 x_0 x_{-1} & \longrightarrow & x_1 x_{-1} x_0 \\
 & \nearrow & & & \searrow \\
 x_0 x_1 x_{-1} & & & & & x_{-1} x_1 x_0 \\
 & \searrow & & & \nearrow \\
 & & x_0 x_{-1} x_1 & \longrightarrow & x_{-1} x_0 x_1
 \end{array}$$

These two ways give the same result.

It is straightforward to verify that the conditions (a) and (b) are satisfied. The covariant algebra defined by (21) and (22) was first introduced by Podleś and interpreted as a noncommutative extension of 2-sphere [6]. The constant  $r$  in (21) may be regarded as square of radius. While the parameter  $\xi$  in (22), which goes to zero in the classical limit, does not exist in a commutative sphere. We thus obtained a one-parameter family of noncommutative 2-spheres.

By taking the higher dimensional corepresentations, one may systematically obtain higher dimensional noncommutative spaces covariant under  $SL_q(2)$ .

## 4 Covariant superspaces of $OSp_q(1|2)$

### 4.1 General prescription

In this section, the discussions in the preceding sections are extended to a quantum supergroup in order to obtain noncommutative superspaces. Since the representation theories of quantum algebra  $U_q[sl(2)]$  and quantum superalgebra  $U_q[osp(1|2)]$  are quite parallel, one can establish a prescription for constructing  $OSp_q(1|2)$ -covariant algebras similar to the one for  $SL_q(2)$  by repeating the same discussion as §3. We give our results without proofs, since the results in this section have already published in [11]. Readers may refer to [11] for details.

The universal enveloping algebra  $\mathcal{U} = U_q[osp(1|2)]$  is generated by the two even  $K^{\pm 1}$ , and the two odd elements  $v_{\pm}$  satisfying the commutation properties [12]

$$\begin{aligned}
 K K^{-1} &= K^{-1} K = 1, & K v_{\pm} &= q^{\pm 1/2} v_{\pm} K, \\
 \{v_+, v_-\} &= -\frac{K^2 - K^{-2}}{q^4 - q^{-4}}.
 \end{aligned} \tag{23}$$

Each irreducible representation (finite dimensional) of the algebra  $\mathcal{U}$  is specified by a nonnegative integer  $\ell$  and the corresponding  $(2\ell + 1)$  dimensional representation space  $V^{(\ell)}$  is also  $\mathbb{Z}_2$  graded. Let  $\{e_m^{\ell}(\lambda) \mid m = \ell, \ell - 1, \dots, -\ell\}$

be a basis of  $V^{(\ell)}$ , where each basis vector has a definite parity. The index  $\lambda = 0, 1$  specifies the parity of the highest weight vector  $e_\ell^\ell(\lambda)$ . The parity of  $e_m^\ell(\lambda)$  equals  $\ell - m + \lambda$ , as it is obtained by the application of  $v_-^{\ell-m}$  on  $e_\ell^\ell(\lambda)$ .

Tensor product of two irreducible representations of  $\mathcal{U}$  has been discussed in [12, 13]. It is, in general, reducible and decomposed into a direct sum of irreducible representations. The rule of decomposition is identical to the classical case:

$$\ell_1 \otimes \ell_2 = \ell_1 + \ell_2 \oplus \ell_1 + \ell_2 \oplus \ell_1 + \ell_2 - 1 \oplus \cdots \oplus |\ell_1 - \ell_2|. \quad (24)$$

The irreducible basis of the tensor product representations is obtained by using the CGC:

$$e_m^\ell(\ell_1, \ell_2, \Lambda) = \sum_{m_1, m_2} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell} e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda), \quad (25)$$

where  $m = m_1 + m_2$ , and  $\Lambda = \ell_1 + \ell_2 + \ell \pmod{2}$  is the parity of the highest weight vector  $e_\ell^\ell(\ell_1, \ell_2, \Lambda)$ . The CGC for the algebra  $\mathcal{U}$  has been computed in [11, 13] and the orthogonality relations similar to (11), (12) have also been obtained.

On the contrary to  $SL_q(2)$ , explicit expressions of corepresentation for  $\mathcal{A} = OSp_q(1/2)$  have not known yet. Employing the duality of the algebras  $\mathcal{U}$  and  $\mathcal{A}$ , one can obtain the hitherto unknown corepresentations of  $\mathcal{A}$  from the already known irreducible representations of  $\mathcal{U}$ . Let  $D^\ell(X; \lambda)$  be a representation matrix of  $X \in \mathcal{U}$  on  $V^{(\ell)}$

$$X e_m^\ell(\lambda) = \sum_{m'} e_{m'}^\ell(\lambda) D_{m'm}^\ell(X; \lambda). \quad (26)$$

We define a corepresentation matrix  $T^{(\ell)}(\lambda)$  of  $\mathcal{A}$  via the duality relation

$$D_{m'm}^\ell(X; \lambda) = (-1)^{\hat{X}(\ell-m'+\lambda)} \left\langle X, T_{m'm}^{(\ell)}(\lambda) \right\rangle, \quad (27)$$

and the parity may be assigned as

$$\widehat{T_{m'm}^{(\ell)}}(\lambda) = m' + m \pmod{2}. \quad (28)$$

With this corepresentation matrix, one can show that  $V^{(\ell)}$  is a right comodule of  $\mathcal{A}$  and that  $T^{(\ell)}(\lambda)$  satisfies the product law similar to (13). It is not difficult to find  $T^{(\ell)}(\lambda)$  for lower values of  $\ell$  from (27). For  $\ell = 1$ , we obtain

$$T^{(1)}(0) = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix}, \quad T^{(1)}(1) = \begin{pmatrix} a & -\alpha & b \\ -\gamma & e & -\beta \\ c & -\delta & d \end{pmatrix}, \quad (29)$$

where the entries in latin (greek) characters are of even (odd) parity. For  $\ell = 2$ ,

the entries of corepresentation matrix are quadratic in  $\ell = 1$  entries

$$T^{(2)}(0) = \begin{pmatrix} a^2 & \kappa_1 a \alpha & \kappa_3 a b & \kappa_1 \alpha b & b^2 \\ \kappa_1 a \gamma & a e + q^{-1} \gamma \alpha & \kappa_2 (a \beta + q^{-1} \gamma b) & -\alpha \beta + q^{-1} e b & \kappa_1 b \beta \\ \kappa_3 a c & \kappa_2 (a \delta + c \alpha) & a d + q^{-1} [2] \alpha \delta + q^{-2} b c & \kappa_2 (\alpha d + \delta b) & \kappa_3 b d \\ \kappa_1 \gamma c & \gamma \delta + q^{-1} c e & \kappa_2 (\gamma d + q^{-1} c \beta) & e d + q^{-1} \beta \delta & \kappa_1 \beta d \\ c^2 & \kappa_1 c \delta & \kappa_3 c d & \kappa_1 \delta d & d^2 \end{pmatrix}, \quad (30)$$

where

$$\kappa_1 = \sqrt{\frac{[4]}{q[2]}}, \quad \kappa_2 = \sqrt{q^{-1}[3]}, \quad \kappa_3 = \kappa_1 \kappa_2, \quad [n] = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{-1/2} + q^{1/2}}. \quad (31)$$

We are ready to discuss covariant algebras for the quantum supergroup  $\mathcal{A}$ . Following the arguments for  $SL_q(2)$ , we define the composite object

$$E_M^L \equiv \mu(e_M^L(\ell, \ell, \Lambda)) = \sum_{m_1, m_2} C_{m_1 m_2 M}^{\ell \ell L} e_{m_1}^\ell(\lambda) e_{m_2}^\ell(\lambda), \quad (32)$$

where  $\Lambda = L \pmod{2}$ . Then the right coaction on  $E_M^L$  is shown to be

$$\varphi_R(E_M^L) = \sum_{M'} E_{M'}^L \otimes T_{M'M}^{(L)}(\Lambda). \quad (33)$$

A following set of covariant relations are extracted from (33) depending on the values of  $L$

$$E_0^0(0) = \sum_{m_1, m_2} C_{m_1 m_2 0}^{\ell \ell 0} e_{m_1}^\ell(\lambda) e_{m_2}^\ell(\lambda) = r, \quad (L = 0) \quad (34)$$

$$E_m^\ell(\lambda) = \sum_{m_1, m_2} C_{m_1 m_2 m}^{\ell \ell \ell} e_{m_1}^\ell(\lambda) e_{m_2}^\ell(\lambda) = \xi e_m^\ell(\lambda), \quad (L = \ell) \quad (35)$$

$$E_M^L = \sum_{m_1, m_2} C_{m_1 m_2 M}^{\ell \ell L} e_{m_1}^\ell(\lambda) e_{m_2}^\ell(\lambda) = 0, \quad (L \neq 0, \ell) \quad (36)$$

In (35), the proportionality constant  $\xi$  is of even parity if  $\lambda = \ell \pmod{2}$ , or odd parity if  $\lambda \neq \ell \pmod{2}$ . As already seen in the case of  $SL_q(2)$ , the simultaneous use of all relations from (34) to (36) gives an inconsistent result. We have to make a choice of appropriate relations defining a covariant algebra. Then we should check the consistency conditions (a) and (b) given in §3.2.

## 4.2 Quantum superspace and quantum superspheres

As an application of the prescription given in the previous subsection, let us examine the covariant algebras corresponding to  $\ell = 1, 2$  with  $\lambda = 0$ . The covariant algebra for  $\ell = 1$  is identified with the quantum superspace. The one for  $\ell = 2$  is interpreted as a noncommutative extension of supersphere.



We start with the case of  $\ell = 1$ , where the relevant tensor product decomposition is given by  $1 \otimes 1 = 2 \oplus 1 \oplus 0$ . We denote the basis of  $V^{(1)}$  by  $z_m = e_m^1(0)$  on which the quantum supermatrix  $T^{(1)}(0)$  in (29) coacts. Thus  $z_{\pm 1}$  are parity even and  $z_0$  is parity odd. Using the CGC given in [11], we obtain from (34) for  $L = 0$

$$q^{1/2}z_{-1}z_1 + z_0^2 - q^{-1/2}z_1z_{-1} = r. \quad (37)$$

For  $L = 1$ , we have  $\Lambda \neq \lambda$ , and, therefore, the parameter  $\xi$  is a Grassmann number:

$$\begin{aligned} -q^{1/2}z_0z_1 + q^{-1/2}z_1z_0 &= \xi z_1, \\ z_{-1}z_1 + (q^{-1/2} + q^{1/2})z_0^2 - z_1z_{-1} &= \xi z_0, \\ q^{1/2}z_{-1}z_0 - q^{-1/2}z_0z_{-1} &= \xi z_{-1}. \end{aligned} \quad (38)$$

For  $L = 2$ , we obtain, using (36), unacceptable relations such as  $z_1^2 = 0$ . We thus take (37) and (38) as defining relations of our covariant algebra. It is not difficult to see that the consistency condition (a) is satisfied, while the condition (b) requires setting  $\xi = 0$ . Therefore, we define our covariant algebra by combining relations (37) and (38), while maintaining  $\xi = 0$ :

$$\begin{aligned} z_1z_0 &= qz_0z_1, & z_0z_{-1} &= qz_{-1}z_0, \\ z_1z_{-1} &= q^2z_{-1}z_1 - q(q^{-1/2} + q^{1/2})r, \\ z_0^2 &= -q^{-1}[2]z_1z_{-1} - q^{-1}r. \end{aligned} \quad (39)$$

This may be interpreted as the most general form of a quantum superspace. The simplest quantum superspace corresponds to the choice of  $r = 0$ .

We next investigate a covariant algebra for  $\ell = 2$ . This may be interpreted as a supersymmetric extension of a noncommutative sphere, since  $\ell = 2$  corresponds to the adjoint representation of the algebra  $\mathcal{A}$ . The quantum supersphere may have applications in integrable quantum field theories. Some models of integrable field theory which has  $osp(1|2)$  symmetry and in which supersphere appears as a target space have been considered [14]. If an extension of such models having quantum algebra symmetry is considered, the quantum supersphere will also appear as a target space.

Let us denote the basis of  $V^{(2)}$  by  $Y_m = e_m^2(0)$ , where  $m = 0, \pm 1, \pm 2$ . Here  $Y_0, Y_{\pm 2}$  are of even parity, and  $Y_{\pm 1}$  are of odd. We seek a covariant algebra under the right coaction of the quantum supermatrix  $T^{(2)}(0)$  in (30). In order to regard the obtained covariant algebra as a noncommutative extension of a supersphere, we need one relation defining the radius of the supersphere and ten commutation relations of supersphere coordinates  $Y_m$ . In addition to those relations, two more relations which relate  $Y_{\pm 1}^2$  to other coordinate are needed, since the odd elements may lose their nilpotency at the quantum level. We thus have to find thirteen relations to define the quantum supersphere.

The relation for radius is obtained via (34), *i.e.*  $L = 0$

$$q^{-1}Y_2Y_{-2} - q^{-1/2}Y_1Y_{-1} - Y_0^2 + q^{1/2}Y_{-1}Y_1 + qY_{-2}Y_2 = r, \quad (40)$$

where  $r$  is a constant corresponding to the square of radius. As commutation relations of the coordinates  $Y_m$ , we admit the sets of covariant relations for  $L = 2$  and  $L = 3$  obtained via (35) and (36). Each of them contains five and seven relations, respectively. We now have obtained the required number of commutation relations and it is easy to verify that their classical limit coincide with the commutative supersphere. To test whether they consistently define an algebra, we need to check for the conditions (a) and (b) mentioned in (3.2). It may be proved by direct computation that the said conditions are, however, not satisfied. In order to make the algebra well-defined, we incorporate the  $L = 1$  relations. With the aid of  $L = 1$  relations, one can verify that the consistency conditions are satisfied. The remaining  $L = 4$  relations can not be incorporated, since they contain unacceptable equations such as  $Y_{\pm 2}^2 = 0$ .

As a result, we have sixteen relations. As all the relations are covariant by construction, their linear combinations are also covariant. Taking linear combinations, the relations which defined the quantum supersphere covariant under the coaction of the algebra  $\mathcal{A}$  are summarized as follows: the radius relation (40), ten commutation relations, two relations for  $Y_{\pm 1}^2$  and three constraints. The classical limit of three constraints are not required in the commutative case. However, we need the constraints to make our algebra well-defined. The relations for  $Y_{\pm 1}^2$  show that the odd coordinate of commutative spheres are no longer nilpotent in the noncommutative setting. Some of the defining relations contains one additional parameter  $\xi$  originated in (35). We thus have obtained a one-parameter family of noncommutative superspheres. Explicit expressions of the defining relations of the quantum supersphere are found in [11].

Before closing this section, we briefly mention some properties of the quantum superspheres obtained above. They enjoy two realizations: The first one is the realization by  $\mathcal{U}$ -covariant oscillator introduced in [15]. This realization allows us, via realizing the covariant oscillator in terms of conventional  $q$ -oscillator, to obtain a infinite dimensional matrix representation of our quantum supersphere. In the second realization, the coordinates  $Y_m$  of quantum superspheres are expressed in terms of the elements of  $\mathcal{A}$ . More precisely,  $Y_k$  is a linear combination of the entries of  $k$ th column of the adjoint corepresentation matrix (30). Therefore, the quantum supersphere can be regarded as a subalgebra of  $\mathcal{A}$ . This subalgebra is specified by the infinitesimal characterization which was first developed for  $SL_q(2)$  [16]. The infinitesimal characterization tells us that amongst subalgebras of  $\mathcal{A}$ , the quantum supersphere is the one annihilated by a linear combination of the twisted primitive elements of  $\mathcal{U}$ . An element  $u \in \mathcal{U}$  possessing a coproduct structure  $\Delta(u) = g \otimes u + u \otimes g^{-1}$  with  $g \in \mathcal{U}$  being a group-like element is said to be twisted primitive with respect to  $g$ . There exist three twisted primitive elements in  $\mathcal{U}$ , that is,  $v_{\pm}$  and  $K - K^{-1}$ . We now define an action of an element of  $u \in \mathcal{U}$  on  $a \in \mathcal{A}$  by

$$a \odot u = (-1)^{\hat{a}\hat{u}}(u \otimes \text{id})(\Delta(a)) = \sum (-1)^{\hat{a}\hat{u}} \langle u, a_{(1)} \rangle a_{(2)}, \quad (41)$$

where Sweedler's notation for coproduct,  $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ , is used and  $\hat{a}, \hat{u}$  denote the parity of the elements  $u, a$ . For a twisted primitive element  $u$ , it

is straightforward to verify that

$$a \odot u = 0 \quad \text{and} \quad b \odot u = 0 \quad \Rightarrow \quad (ab) \odot u = 0. \quad (42)$$

Thus a set of elements of  $\mathcal{A}$  annihilated by a twisted primitive element  $u$  form a subalgebra of  $\mathcal{A}$ . Indeed, the quantum supersphere realized in terms of  $T^{(2)}$  in (30) is a subalgebra of  $\mathcal{A}$  that is annihilated by the twisted primitive element  $\mathcal{P}_R$

$$\mathcal{P}_R = -\sqrt{g_3} v_+ + \sqrt{g_1} v_-, \quad (43)$$

$$Y_k \odot \mathcal{P}_R = 0, \quad k = \pm 2, \pm 1, 0. \quad (44)$$

$\mathcal{P}_R$  consists of only odd twisted primitive elements. This is a difference from the quantum sphere for  $SL_q(2)$ . In that example, all the twisted primitive elements contribute to the annihilation operator of quantum sphere.

## 5 Concluding remarks

We have developed a common general prescription for constructing noncommutative covariant spaces of  $SL_q(2)$  and  $OSp_q(1|2)$ . By this construction, it is possible to obtain covariant algebras for a given representation of  $SL_q(2)$  or  $OSp_q(1|2)$ . Indeed, the known noncommutative spaces, namely Manin's quantum plane and Podleś' quantum spheres, were recovered for  $SL_q(2)$  and novel ones, *i.e.* their extensions to  $OSp_q(1|2)$ , were obtained. We note a difference of the present work from others [1, 18, 19]. In order to obtain higher dimensional noncommutative spaces, we use a higher dimensional representation of the fixed quantum group  $SL_q(2)$  (or  $OSp_q(1|2)$ ), while higher rank quantum groups are used in [1, 18, 19].

We believe that the results of this work are useful for making progress in constructing supersymmetric versions of noncommutative geometry. For instance, we construct noncommutative superspace, say quantum supersphere, by our method. Then we may consider differential calculi on the space. It allows us to compute its curvature, metric and so on based on the framework of Ref. [17]. It may also be possible to extend our method to higher rank quantum (super) groups by taking into account the multiplicity of irreducible decomposition of tensor product representations.

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